Conformal blocks

P.I. pre talk 5/22/24.

"Abstract"
Conformal blocks for affine Kac-Moody Lie algebras are closely related to the infinitesimal structure for Bun_G. Here, we define conformal blocks as a certain vector bundle over the configuration space $\mathbb{C}^n \setminus \{x_i = x_j\}$. & explain how to construct a flat connection on it.

Let $\mathfrak{g}$ = fin. dim simple Lie alg $/\mathbb{C}$

$X = \mathbb{P}^1$, Fix $\infty \in \mathbb{P}^1$ w/ coordinate t on $\mathbb{P}^1 \setminus \infty$.

$\mathfrak{g}((t)) := \mathfrak{g} \otimes \mathbb{C}((t))$ = "loop algebra"

$\hat{\mathfrak{g}}$:= affine Kac-Moody algebra, defined as central extension

$$0 \longrightarrow \mathbb{C}K \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g}((t)) \longrightarrow 0$$

So, $[A \otimes f, B \otimes g] = [A,B] \otimes fg - \mathrm{Res}_{t=0}(f\,dg)\,(A,B)\cdot K$ ← killing form on $\mathfrak{g}$.

← defines Lie bracket on $\hat{\mathfrak{g}}$.

Is $\mathbb{Z}$-graded vertex algebra

$$V_k\,\mathfrak{g} := \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}_k = \text{Vacuum rep of level } k$$

(K acts by k, $\mathfrak{g}[[t]]$ acts by 0)

PBW $\Rightarrow$ $\hat{\mathfrak{g}} = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus (\mathfrak{g}[[t]] \oplus k)$

$V_k(\mathfrak{g}) \simeq U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$

Let $\vec{x} = (x_1, \dots, x_N) \in U_N := \mathbb{C}^N \setminus \bigcup \{x_i = x_j\}$

$L\mathfrak{g}(x_i) := \mathfrak{g}((t - x_i))$ "loop algebra around x_i"

$\rightsquigarrow$ diagonal central extension

$$0 \longrightarrow \mathbb{C} \longrightarrow \hat{\mathfrak{g}}(\vec{x}) \longrightarrow \bigoplus_{i=1}^{N} L\mathfrak{g}(x_i) \longrightarrow 0$$

Let $M_1, \dots, M_N$ be $\mathfrak{g}$-reps & define

$$\mathbb{M}_i := \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}[K]}^{\hat{\mathfrak{g}}} M_i = \hat{\mathfrak{g}}\text{-mod of level } k$$

K acts by k

$\mathbb{M} := \bigotimes_{i=1}^{N} \mathbb{M}_i$

($M_i = \mathfrak{g} \otimes t\mathbb{C}[[t]]$ -invariants of $\mathbb{M}_i$)

Let $\mathfrak{g}^{out}_{\vec{x}} := \mathfrak{g} \otimes \mathbb{C}[\mathbb{P}^1 \setminus \{x_1, \dots, x_n\}] \xhookrightarrow[\text{expand around } x_i]{\text{Taylor}} \bigoplus_{i=1}^{N} L\mathfrak{g}(x_i)$

Def The space of conformal blocks is

$$C(\mathbb{P}^1, (x_i), (\mathbb{M}_i))_{i=1}^{N} := \mathrm{Hom}_{\mathfrak{g}^{out}_{\vec{x}}}(\mathbb{M}, \mathbb{C})$$

Def the space of coinvariants $\mathbb{M}^{out}_{\vec{x}} := \mathbb{M} / \mathfrak{g}^{out}_{\vec{x}} \cdot \mathbb{M} \simeq$ dual of conformal blocks

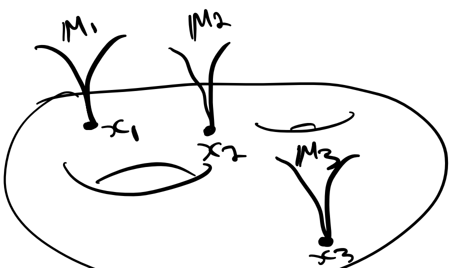
Next, by varying the points, we get a bundle of conformal blocks,

$$\begin{array}{c} \mathcal{E} = C(\mathbb{P}^1, \underline{x}, \hat{\otimes}_i \mathbb{M}_i) \\ \downarrow \\ U_n = \mathbb{A}^N \setminus \{x_i = x_j\} \end{array}$$

Fiber over $\vec{x} \in U_n$ is conformal block $(\mathbb{M}^{out}_{\vec{x}})^*$.

Property: $\mathbb{M}^{out}_{\vec{x}} \simeq \mathbb{M}^{out}_{\vec{x}'}$ for $\vec{x}'$ nearby $\vec{x}$.

Remarks (1) We may define $C_V(X, (x_i), (\mathbb{M}_i))$ for any vertex algebra V, proj. curve X, V-modules $\mathbb{M}_i$ over points x_i using chiral correlation functions studied in 2d CFT:



A conformal block is a functional / correlator

$$\varphi : \bigotimes \mathcal{M}_{i,x_i} \longrightarrow \mathbb{C}$$

which is $U^{out}_{\vec{x}}(V)$-invariant,

where $\mathcal{M}_i$ are vector bundles on X associated to $\mathbb{M}_i$ = V-modules

$\mathcal{M}_i = \mathrm{Aut}_X \times_{\mathrm{Aut}_O} \mathbb{M}_i$ ← The twist amounts to choosing formal coord. around each $x_i \in X$, but we may construct conformal blocks w/o this.

(2) Elements of $\mathcal{M}_{i,x_i}$ are observables, and φ is a correlator which takes the product of observables & outputs an expected value.

(3) Vacuum insertion asserts adding a vacuum module at new point does not change the conformal block. (Explain "1 point uniformization of $\mathrm{Bun}_G \simeq$ 2 point uniformization.")

(4) Conformal blocks satisfy factorization properties.

Witten conjecture: $\exists$ flat (projective) connection on the bundle $\mathcal{E}$ of conformal blocks over U_N. First proved by Hitchin.

Locally, claim says $\exists\ \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_{U_n}$ such that over $(x_1, \ldots, x_n)$

$$\begin{cases} \nabla_i = \frac{\partial}{\partial x_i} + A_i, & A_i \text{ act on fibers of } \mathcal{E}, \text{ (connection)} \\ [\nabla_i, \nabla_j] = 0 & \text{(flat).} \end{cases}$$

[projective means $[\nabla_i, \nabla_j] = \lambda \cdot Id$, $\lambda \in \mathbb{C}$, and A defined up to adding a scalar matrix)

Thus, $(\mathcal{E}, \nabla)$ gives a local system over U_n, & consequently there's a monodromy rep of $\pi_1(U_n) = $ Braid group on the fibers.

"Casimir" $=: \Omega_{ij}$

Easy Fact $\nabla_i = \frac{\partial}{\partial x_i} - \frac{1}{\underset{\text{level}}{k} + \underset{\text{Coxeter \#}}{h^\vee}} \left[\sum_{j \neq i} \frac{\sum_a J_a^{(i)} J^{a(j)}}{x_i - x_j} \right]$,

where $J_a^{(i)} := J_a \otimes z^i$, $\{J_1, \ldots, J_{\dim \mathfrak{g}}\}$ are basis of $\mathfrak{g}$

$\Longrightarrow$ Horizontal sections of $\mathcal{E}$ are functions

$$\Phi : U_N \longrightarrow M_1^* \otimes \cdots \otimes M_N^*$$

(so, work over $\mathbb{P}^1$ instead of $\mathbb{C}^1$)

Rmk: If want horizontal sections for the projective flat connection, then these Φ must land in $\mathfrak{g}$-invariants $= \mathrm{Hom}_{\mathfrak{g}}(M_1 \otimes \cdots \otimes M_n, \mathbb{C})$.

which satisfy

$$\boxed{(k + h^\vee) \frac{\partial}{\partial x_i} \Phi = \left[\sum_{j \neq i} \frac{\Omega_{ij}}{x_i - x_j} \right] \Phi}$$

. These are called the Knizhnik - Zamolodchikov (KZ) equations.

The KZ system is holonomic, consistent, with regular singularities along $x_i = x_j$

. Solutions are known explicitly since $X = \mathbb{P}^1$. (Also explored for elliptic curves, but not done for higher genus)

Asymptotic Regions: For $\sigma \in S_N$, let

simply-connected $D_\sigma := \{ z \in \mathbb{R}^N \mid z_{\sigma(1)} > z_{\sigma(2)} > \cdots > z_{\sigma(N)} > 0 \}$

Then, $\underset{\substack{\nabla \\ \text{horizontal sections}}}{\Gamma}(D_\sigma, \mathcal{E}) \cong (M^{\text{out}}_{\vec{x}})^* \simeq \begin{cases} M_1^* \otimes \cdots \otimes M_N^* & \text{over } \mathbb{A}^1 \text{ (require vanish at } \infty) \\ (M_1^* \otimes \cdots \otimes M_n^*)^{\mathfrak{g}} & \text{over } \mathbb{P}^1 \end{cases} =: M$

We keep track of these isomorphisms:

$$u_\sigma : D_\sigma \xrightarrow[\text{diffeo}]{\sim} \mathbb{R} \times (\mathbb{R}_{>0})^{N-1} \rightsquigarrow u_\sigma : U_n \xrightarrow[\text{biholom}]{\sim} \mathbb{C} \times (\mathbb{C}^\times)^{N-1}$$

$u_\sigma : (z_1, \dots, z_n) \longmapsto (z_1 + \dots + z_n, u_i)$, $u_1 = z_1 - z_2$, $u_2 = \frac{z_2 - z_3}{z_1 - z_2}$, $\dots$, $u_{N-1} = \frac{z_{N-1} - z_N}{z_{N-2} - z_{N-1}}$)

$\rightsquigarrow \Psi_\sigma : \Gamma_\nabla(D_\sigma, \mathcal{E}) \xrightarrow{\sim} M$, called "asymptotics of a solution to KZ equation in the limit $z_{\sigma(1)} \gg \dots \gg z_{\sigma(N)}$"

Comment: $\frac{1}{z_1 - z_2}$ has 2 different expansions as power series - one corresponds to region where $z_1 \gg z_2$ & other to $z_1 \ll z_2$.

So, taking "asymptotics of a solution" means search for solutions of KZ eqn with the specified power series expansion.

Fix base point $z \in D_{id}$. Let $\gamma^\pm : z \to \sigma(z)$ be unique (up to isotopy) path connecting $z, \sigma(z)$ s.t. every z_i passes above (γ^+) or below (γ^-) z_j if $j > i$

Def $M_\sigma^\pm(\infty) : M \xrightarrow{\sim} \Gamma_\nabla(D_{id}, \mathcal{E}) \xrightarrow{\gamma^\pm} \Gamma_\nabla(D_\sigma, \mathcal{E}) \xrightarrow{\sim} M$

asymptote at $z_1 \gg \dots \gg z_N$ asymptote at $z_{\sigma(1)} \gg \dots \gg z_{\sigma(N)}$.

$\cdot\, M_\sigma^+ \circ M_{\bar\sigma}^- = \mathrm{Id}$.

Thm [Drinfeld-Kohno]

- $M_\sigma^{\pm 1}(\infty)$ satisfy braid relations, defining a $\pi_1(U_n, z)$-rep on M.
- $M_\sigma^{\pm 1}(\infty)$ define a symmetric monoidal structure on $\mathrm{Rep}(\mathfrak{g})$ & $\mathrm{Rep}\,\mathfrak{g} \simeq \mathrm{Rep}(U_q \mathfrak{g})$ for $q^n \neq 1$ as $\otimes$-cats.

Rmk (1) The $\{\Omega_{ij}\}$ appearing in KZ eqn satisfy $[\Omega_{ij}, \Omega_{kl}] = 0$ & $[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0$ & the free Lie alg subject to these relations is called the Drinfeld-Kohno Lie alg.

(2) The Verlinde formula computes the rank of $\mathcal{E}$.

(3) We may refine the above & ask for solutions to KZ landing in $M^{sing} := (M_1^* \otimes \dots \otimes M_N^*)^{\mathfrak{n}^+}$. Then fundamental domain still D_σ, but now we have $\Psi_{\sigma, T} : \Gamma(D_\sigma, \mathcal{E})^{sing} \longrightarrow M^{sing}$ for $[N]$-trees T.

monodromy between KZ solutions parallelized by

Then $\boxed{\left(\pi_1(U_n)\text{-rep on } M^{sing} \text{ given by KZ}\right) \simeq \left(\text{Universal } R\text{-matrix for } U_q \mathfrak{g}\right)}$ ($\exists\, q : q^n \neq 1$ version)

Now, we give 2 motivations for our main result:

1) We saw the asymptotic regions of solutions to KZ equations are parameterized by $[n]$-trees. There is a Fulton-Macpherson compactification of U_n, denoted $\hat{\mathbb{P}}^{[n]}$, for which strata are also parameterized by $[n]$-trees, & the diagonal divisor ~~becomes~~ normal crossing. One hope is to extend the flat connection

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\nabla"'] & & \hat{\mathcal{E}} \arrow[d] \\ U_n = \mathbb{A}^n \setminus \Delta \arrow[rr, hook] & & \hat{\mathbb{P}}^{[n]} \end{tikzcd}$$

Does there exist flat connection $\hat{\nabla}$ here?

<u>Answer:</u> Yes! This is done e.g. in [Tsuchiya-Ueno-Yamada] & the connection ∇ extends to a flat, projective connection $\hat{\nabla}$ which is regular on $\hat{\mathbb{P}}^{[n]} \setminus \{x_i = x_j\}$, and with log-singularities on diagonal divisor $\hat{D}$.

In fact, they extend:

determinant line bundle on $Bun_G(\mathcal{X}) = \{(P, X) = G\text{-bundle, curve}\}$

$$\begin{tikzcd} \text{conformal blocks} = H^0(Bun_G(\mathcal{X}), \det^{\otimes k}) =: \mathcal{E} \arrow[d, "\nabla \text{ flat}"] \arrow[rr, squiggly] & & \hat{\mathcal{E}} \arrow[d] \\ M_{g,n} \arrow[rr, hook, "\text{Deligne-Mumford compactification}"'] & & \overline{M}_{g,n} \end{tikzcd}$$

$\hat{\nabla}$ flat proj. connection w/ log singularities on the boundary.

2) Relation to <u>Geometric Langlands</u>

We may modify the bundle of coinvariants (or, dually, conformal blocks) to obtain a bundle (central charge $k=0$)

$$\begin{tikzcd} \Delta(V_0(\mathfrak{g})) \arrow[d] \\ Bun_G(X) \end{tikzcd}$$

with fiber at P = "P-twisted coinvariants"

associated P-bundle

$$= V_{k=0}(\mathfrak{g})^P_x \big/ \mathfrak{g}^P_{out}(x) \cdot V_{k=0}(\mathfrak{g})^P.$$

which comes from the "localization functor of GLC":

$$\Delta : (\mathfrak{g}_{K_x}, G(\mathcal{O}_x))\text{-mod} \xrightarrow{\text{localization}} D_{\mathrm{Bun}_G}\text{-mod}$$

$$V \longmapsto \left(\pi_*\left(D_{\mathrm{Bun}_G^{\mathrm{lev}}} \otimes_{U\mathfrak{g}_{K_x}} V\right)\right)^{G(\mathcal{O}_x)} \quad \text{as } \mathcal{O}_{\mathrm{Bun}_G}\text{-module}$$

"Quantum Hamiltonian Reduction"

where $\pi : \mathrm{Bun}_G^{\mathrm{lev}}(X) \longrightarrow \mathrm{Bun}_G(X)$

$\mathrm{Bun}_G^{\mathrm{lev}}(X) = G(\mathcal{O}_{\mathrm{out}})\backslash G(K_x)$, $\quad \mathrm{Bun}_G(X) = G(\mathcal{O}_{\mathrm{out}})\backslash G(K)/G(\mathcal{O})$

Then it is known $\Delta(V_k(\mathfrak{g}))$ over $\mathrm{Bun}_G(X)$ also admits a flat connection (e.g. Nick Rozenblyum's thesis), consequently it is a D_{Bun_G}-module. One may hope that there's an explicit construction of this connection using the description of infinitesimal jet spaces that I'll explain tomorrow. Evidence of this is that Ginzburg did this for the analogous bundle of conformal blocks on $\mathcal{M}_g$ = moduli space of genus g curves